1. Rigid body modes

Calculation of rigid body properties

This section discusses the theory used in the calculation of rigid body properties. Experimental frequency response functions (FRFs) can be used to derive structural modes of a structure and the inertia properties of a system.

These properties are:

- the moments of inertia
- the products of inertia
- the principal moments of inertia

In general two types of method are applied:

1. A first type determines the inertia characteristics using the rigid body mode shapes obtained from test data. This is the Modal Model Method described in reference [1].

2. The second type starts from the mass line, i.e. the FRF inertia restraint of the softly suspended structure. This mass line is used in a set of kinematic and dynamic equations, from which the rigid body characteristics (mass, center of gravity, principal directions and moments of inertia) can be determined (reference [2]).

   Some of these methods also look for the suspension stiffnesses while others consider the mass of the system as known (reference [3]).

   This type of method is described in more detail below.

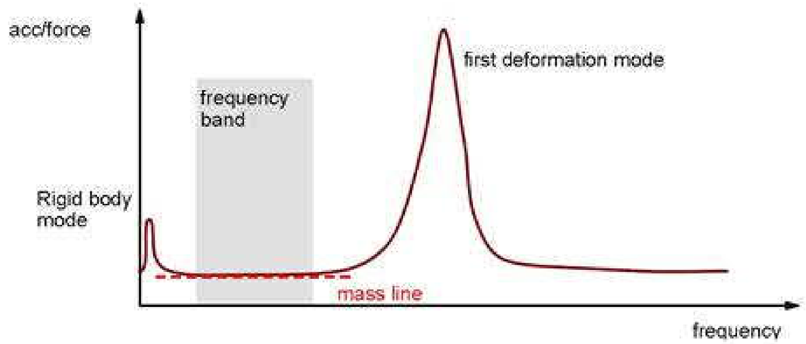

Figure 1-1. Rigid body modes

Derivation of rigid body properties from measured FRFs

Input data

FRFs are required in order to determine the rigid body properties. The input format is required to be acceleration/force, and if this is not the case then a transformation can be applied. Rotational or scalar (acoustic) measurements are not used in the rigid body calculations.

In theory 2 excitations and 6 responses are needed for the calculations. Practical tests show that best results are obtained with at least 6 excitations (e.g. 2 nodes in 3 directions) and 12 responses need to be measured.

Reference axis system

All the rigid body properties are calculated relative to a reference axis. The reference axis system is defined by the three coordinate values of its origin and three euler angles representing its rotation.

Specification of the frequency band

Rigid body properties are calculated in a global (least squares) sense over a specified frequency band between the last rigid body mode and the first deformation mode (see figure **Rigid body modes**).

Mass line value

The "mass line" value which is needed for the calculations, can be derived from the measured FRFs in three ways:

1. If the rigid body modes and deformation modes are sufficiently spaced, the amplitude values (with the sign of the real part) of the original, unchanged measured FRFs can be used. In this case there is no need to have the deformation modes available for the rigid body modes analysis.

2. If the spacing between rigid body modes and deformation modes is not sufficient, the FRFs have to be corrected. In this case the influence of the first deformation modes, if significant, can be subtracted from the original FRFs. The amplitude values (with the sign of the real part) of *synthesized FRFs* are used.

3. If accurate measured FRFs are not available in the frequency range directly above the rigid body modes, then lower residual terms which lie in a frequency band which contains the first deformation modes can be used. Residual terms can be determined from a modal analysis. Lower residuals represent the influence of the modes *below* the deformation modes and are therefore representative of the rigid body modes.

Calculation of the rigid body properties

Calculation of the reference acceleration matrix

Coordinate transformation

If nodes, corresponding to the response DOFs used do *not* have global directions or when a reference (not coincident with the global origin) is specified, then a rotation of the measured accelerations according to the global/reference axis system is needed.

All three directions `(+X,+Y,+Z)` are required. For the three measured (local) accelerations of output node "o":

$$\{\ddot{X}\}_g = [T]_o^{-1}\,\{\ddot{X}\}_l \qquad \text{Eqn 6-1}$$

where:

$\{X\}_g$	the global acceleration vector
$\{X\}_l$	the local acceleration vector
$[T]_o^{-1}$	the rotation matrix (global to local) of node "o"

If a reference is specified, which does not coincide with the global origin, the three measured accelerations of output node "o" are also rotated according to the axis of the reference system.

$$\{\ddot{X}\}_r = ([T]_o^{-1}\,[T]_r)\,\{\ddot{X}\}_l \qquad \text{Eqn 6-2}$$

where:

$[T]_r$	the rotation matrix (global to local) of node "r"

System of equations

For all spectral lines of the selected band, for all response nodes *P*, *Q*,... and for all inputs `1,2,...` under consideration

$$\begin{bmatrix} \ddot{X}_{1P_x} & \ddot{X}_{2P_x} & \cdots \\ \ddot{X}_{1P_y} & \ddot{X}_{2P_y} & \cdots \\ \ddot{X}_{1P_z} & \ddot{X}_{2P_z} & \cdots \\ \ddot{X}_{1Q_x} & \ddot{X}_{2Q_x} & \cdots \\ \ddot{X}_{1Q_y} & \ddot{X}_{2Q_y} & \cdots \\ \ddot{X}_{1Q_z} & \ddot{X}_{2Q_z} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & Z_P & -Y_P \\ 0 & 1 & 0 & -Z_P & 0 & X_P \\ 0 & 0 & 1 & Y_P & -X_P & 0 \\ 1 & 0 & 0 & 0 & Z_Q & -Y_Q \\ 0 & 1 & 0 & -Z_Q & 0 & X_Q \\ 0 & 0 & 1 & Y_Q & -X_Q & -0 \\ & & & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \ddot{X}_{1g_x} & \ddot{X}_{2g_x} & \cdots \\ \ddot{X}_{1g_y} & \ddot{X}_{2g_y} & \cdots \\ \ddot{X}_{1g_z} & \ddot{X}_{2g_z} & \cdots \\ \ddot{\alpha}_{1g_x} & \ddot{\alpha}_{2g_x} & \cdots \\ \ddot{\alpha}_{1g_y} & \ddot{\alpha}_{2g_y} & \cdots \\ \ddot{\alpha}_{1g_z} & \ddot{\alpha}_{2g_z} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \qquad \text{Eqn 6-3}$$

acceleration of input 1 towards global axis system where:

X_P, Y_P, Z_P	the global coordinates of node *P* (or towards the reference axis system)

This over-determined system of equations (number of output DOFs is higher than or equals 6) is solved for each spectral line in a least square sense. In this way at each spectral line, the reference acceleration matrix is found. Further, a general solution of the reference acceleration matrix over the total frequency band is calculated by solving in a least squares sense the global set of equations containing all outputs and all spectral lines.

Calculation of the reference force matrix

Coordinate transformation

For input force $\{F_1\}$ in the local *X*-direction of node "i":

$$\{F_1\} = [T]_i^{-1}\begin{Bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{Bmatrix} \qquad \text{Eqn 6-4}$$

where:

$[T]_i^{-1}$	the rotation matrix (global to local) of node "i"

If the reference "r" is not coincident with the global origin:

$$\{F_1\} = ([T]_r[T]_i^{-1})\begin{Bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{Bmatrix} \qquad \text{Eqn 6-5}$$

where:

$[T]_r^{-1}$	the rotation matrix (global to local) of reference node "r"

Similar equations are used when the input has *Y*-direction or *Z*-direction.

System of equations:

For all inputs `1,2,...`:

$$\begin{Bmatrix} F_{1g_x} \\ F_{1g_y} \\ F_{1g_z} \\ M_{1g_x} \\ M_{1g_y} \\ M_{1g_z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -Z_1 & Y_1 \\ Z_1 & 0 & -X_1 \\ -Y_1 & X_1 & 0 \end{bmatrix} \{F_1\} \qquad \text{Eqn 6-6}$$

reference force matrix towards global axis system for input 1

where:

{F_1}	the applied force at input 1
X_1, Y_1, Z_1	the global coordinates of node corresponding with input 1

Calculation of the co-ordinates of the center of gravity and moments and products of inertia

For:

(i) each input and each spectral line

(ii) each input over the total band:

$$\begin{Bmatrix} F_{g_x} - m.a_{g_x} \\ F_{g_y} - m.a_{g_y} \\ F_{g_z} - m.a_{g_z} \\ M_{g_x} \\ M_{g_y} \\ M_{g_z} \end{Bmatrix} = \begin{bmatrix} 0 & -ma_z & ma_y & 0 & 0 & 0 & 0 & 0 & 0 \\ ma_z & 0 & -ma_x & 0 & 0 & 0 & 0 & 0 & 0 \\ -ma_y & ma_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{g_z} & -F_{g_y} & a_x & 0 & 0 & -a_y & 0 & -a_z \\ -F_{g_z} & 0 & F_{g_x} & 0 & a_y & 0 & -a_x & -a_z & 0 \\ F_{g_y} & -F_{g_x} & 0 & 0 & 0 & a_z & 0 & -a_y & -a_x \end{bmatrix} \begin{Bmatrix} X_{cog} \\ Y_{cog} \\ Z_{cog} \\ I_{xx} \\ I_{yy} \\ I_{zz} \\ I_{xy} \\ I_{yz} \\ I_{xz} \end{Bmatrix} \qquad \text{Eqn 6-7}$$

where:

X_{cog}, Y_{cog}, Z_{cog}	the global *coordinates of the center of gravity*
I_{xx}, I_{yy}, I_{zz}	the *moments of inertia* towards the global axis system
I_{xy}, I_{yz}, I_{xz}	the *products of inertia* towards the global axis system

This set of equations can be solved in two steps. First, the coordinates of the center of gravity can be solved from the first three equations (per reference). Afterwards, these values can be filled in the last equations to solve the inertia moments and products.

1. For each input and for each spectral line and for each input over the total band:

$$\begin{Bmatrix} F_{g_x} - m a_{g_x} \\ F_{g_y} - m a_{g_y} \\ F_{g_z} - m a_{g_z} \end{Bmatrix} = \begin{bmatrix} 0 & -m a_z & m a_y \\ m a_z & 0 & -m a_x \\ -m a_y & m a_x & 0 \end{bmatrix} \begin{Bmatrix} x_{cog} \\ y_{cog} \\ z_{cog} \end{Bmatrix} \qquad \text{Eqn 6-8}$$

2. For each input and for each spectral line and for each input over the total band:

$$\begin{Bmatrix} M_{g_x} - y_{cog} F_{g_z} + z_{cog} F_{g_y} \\ M_{g_y} + x_{cog} F_{g_z} - z_{cog} F_{g_x} \\ M_{g_z} - x_{cog} F_{g_y} + y_{cog} F_{g_x} \end{Bmatrix} = \begin{bmatrix} a_x & 0 & 0 & -a_y & 0 & -a_z \\ 0 & a_y & 0 & -a_x & -a_z & 0 \\ 0 & 0 & a_z & 0 & -a_y & -a_x \end{bmatrix} \begin{Bmatrix} I_{xx} \\ I_{yy} \\ I_{zz} \\ I_{xy} \\ I_{yz} \\ I_{xz} \end{Bmatrix} \qquad \text{Eqn 6-9}$$

At each spectral line, these over-determined sets of equations (number of inputs larger than or equal to 2) are solved in a Least Square sense. A global solution for these rigid body properties over the total band can be found out of the global acceleration matrix over the total frequency band too (see *equation 6-3*). If wanted, only the second set of equations is solved.

In this case the coordinates of the center of gravity are presumed to be known and specified by the user.

In general: $\{L_g\} = [A]\{\omega_g\}$

$$\begin{Bmatrix} L_x \\ L_y \\ L_z \end{Bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \qquad \text{Eqn 6-10}$$

where:

$\{L_g\}$	the vector of total impulse towards the global (reference) axis system
$[A]$	the matrix of inertia (symmetrical)
$\{\omega_g\}$	the vector of velocity

This is an eigenvalue problem, where:

Eigenvalues I_1, I_2, I_3	the 3 principal moments of inertia

Eigenvectors **$\{e_1\}$, $\{e_2\}$, $\{e_3\}$** the directions of the 3 principal axes of inertia

Rigid body mode analysis

A rigid body is a (part of a) structure that does not deform of itself, but that moves periodically as a whole at a certain frequency.

The modal parameters for such a rigid body mode are determined not by the dynamics of the structure itself, but by the dynamic properties of the boundary conditions of that structure. This includes the way it is attached to its surroundings (or the rest of the structure), the stiffness and damping characteristics of suspending elements, its global mass, etc ... . A rigid body can be compared to a simple system with a mass attached to a fixed point by a spring and a damper element.

It has 6 modes of vibration, i.e. translation along the *X*, *Y*, and *Z* axes, and rotation about these axes. Every mode which is measured for such a system will be a linear combination of these 6 modes. Chapter **Derivation of rigid body properties from measured FRFs** describes how it is possible to calculate the inertia properties of a structure based on measured FRFs. This enables you to calculate the center of gravity, moments of inertia and the principle axes as well as synthesized rigid body modes.

This chapter discusses how rigid body modes are used and describes two methods by which the modes can be determined, namely:

- decomposition of measured modes into rigid body modes

- synthesis of rigid body modes based on geometrical data

Use of rigid body analysis

In modal analysis applications, the fact that (part of) a structure acts as a rigid body up to a certain frequency can be used in different ways.

1. *Debugging the measurement set-up*

   Rigid body modes can be used to verify the measurement set-up if the frequency range of measured FRFs covers a rigid body of the entire structure in its suspension (elastic cords or air bags for example). In this case, a simple peak picking procedure and an animation of the resulting mode will indicate which measurement points are not moving "in line" with the rest of the structure.

   Deviations from this rigid body motion can be caused by:

   - non-measured nodes (not moving at all)
   - wrong response point identification (moving out of line)
   - wrong response direction (moving in opposite direction)
   - bad transducers or wrong calibration values (wrong amplitude)
   - other measurement errors